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EXACT SOLUTION OF THE EXTERIOR FUNDAMENTAL MIXED PROBLEM

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An exact solution of the fundamental exterior mixed problem with a circular line of separation of the boundary conditions for a transversely isotropic half-space is proposed. The interior fundamental mixed problem for an isotropic half-space has been examined in [1, 2].

1. Let us consider a transversely isotropic half-space $z \geqslant 0$ whose planes of isotropy are parallel to the boundary. We understand the problem with the following conditions on the boundary $z = 0$:

$$\begin{aligned} &\sigma_z = \sigma(\rho, \varphi), \qquad \tau_{zx} = \tau_{zx}(\rho, \varphi) \\ &\tau_{yz} = \tau_{yz}(\rho, \varphi) \qquad (\rho \leqslant a) \\ &w = w(\rho, \varphi), \qquad u_x = u_x(\rho, \varphi) \\ &u_y = u_y(\rho, \varphi) \qquad (\rho > a) \end{aligned}$$

to be the exterior fundamental mixed problem.

We introduce the complex tangential displacements $u = u_x + iu_y$ and shear stresses $\tau = \tau_{zx} + i\tau_{yz}$, $\overline{\tau} = \tau_{zx} - i\tau_{yz}$. If the requirement of decomposability in Fourier series in the angular coordinate is imposed on the given and desired functions, then by using the method in [3] we reduce the problem of determining the stresses outside the circle $\rho \geqslant a$ to the infinite system of integral equations (1.1)

$$2\rho^{n+1}\int_{\rho}^{\infty}\frac{dx}{x^{2n+2}\sqrt{x^2-\rho^2}}\int_{a}^{x}\frac{G_1 s^2\tau_{n+1}(s)+G_2[2nx^2-(2n+1)s^2]\bar{\tau}_{-n+1}(s)}{\sqrt{x^2-s^2}}s^n ds - \tag{1.1}$$

$$2\pi H\alpha\rho^{-n-1}\int_{a}^{\rho}\sigma_n(x)x^{n+1}dx = F_{n+1}(\rho) \quad (n\geqslant 0)$$

$$2\rho^{n-1}\int_{\rho}^{\infty}\frac{dx}{x^{2n}\sqrt{x^2-\rho^2}}\int_{a}^{x}\frac{G_1 x^2\tau_{-n+1}(s)+G_2[(2n-1)x^2-2n\rho^2]\bar{\tau}_{n+1}(s)}{\sqrt{x^2-s^2}}s^n ds +$$

$$2\pi H\alpha\rho^{n-1}\int_{\rho}^{\infty}\sigma_{-n}(x)x^{-n+1}dx = F_{-n+1}(\rho) \quad (n\geqslant 1)$$

$$4H\rho^n\int_{\rho}^{\infty}\frac{dx}{x^{2n}\sqrt{x^2-\rho^2}}\int_{a}^{x}\frac{\sigma_n(s)e^{in\varphi}+\sigma_{-n}(s)e^{-in\varphi}}{\sqrt{x^2-s^2}}s^{n+1}ds +$$

$$2\pi H\alpha\,\mathrm{Re}\left\{e^{-in\varphi}\rho^{-n}\int_{a}^{\rho}\tau_{-n+1}(x)x^n dx -\right.$$

$$\left.e^{in\varphi}\rho^n\int_{\rho}^{\infty}\tau_{n+1}(x)x^{-n}dx\right\} = \mathrm{Re}\,\{e^{in\varphi}\Phi_n(\rho)\}$$

Here

$$F_{n+1}(\rho) = u_{n+1}(\rho) + 2\pi H\alpha\rho^{-n-1}\int_{0}^{a}\sigma_n(x)x^{n+1}dx -$$

$$\frac{2}{\rho^{n+1}}\int_{0}^{a}\frac{x^{2n}dx}{\sqrt{\rho^2-x^2}}\int_{x}^{a}\frac{G_1 x^2\tau_{n+1}(s)+G_2[2n\rho^2-(2n+1)x^2]\bar{\tau}_{-n+1}(s)}{s^n\sqrt{s^2-x^2}}ds$$

$$F_{-n+1}(\rho) = u_{-n+1}(\rho) -$$
$$\frac{2}{\rho^{n-1}}\int_{0}^{a}\frac{x^{2n-2}dx}{\sqrt{\rho^2-x^2}}\int_{x}^{a}\frac{G_1 s^2\tau_{-n+1}(s)+G_2[(2n-1)s^2-2nx^2]\bar{\tau}_{n+1}(s)}{s^n\sqrt{s^2-x^2}}ds$$

$$\mathrm{Re}\,\{\Phi_n(\rho)e^{in\varphi}\} = w_n(\rho)e^{in\varphi} + w_{-n}(\rho)e^{-in\varphi} -$$
$$\frac{4H}{\rho^n}\int_{0}^{a}\frac{x^{2n}dx}{\sqrt{\rho^2-x^2}}\int_{x}^{a}\frac{\sigma_n(s)e^{in\varphi}+\sigma_{-n}(s)e^{-in\varphi}}{s^{n-1}\sqrt{s^2-x^2}}ds -$$
$$2\pi H\alpha\,\mathrm{Re}\left\{e^{-in\varphi}\rho^{-n}\int_{0}^{a}\tau_{-n+1}(x)x^n dx\right\}$$

$$G_1 = \beta + \gamma_1\gamma_2 H, \qquad G_2 = \beta - \gamma_1\gamma_2 H$$

The constants α, β, γ_1, γ_2, H are determined by the elastic characteristics of the material of the half-space [3]. The complex functions F_k and Φ_k are known from the condition of the problem. The quantities U_k, σ_k, τ_k are coefficients of the Fourier series expansions of the corresponding functions in the angular coordinate. The axisymmetric exterior problem corresponds to $n = 0$. Its detailed solution is presented in [4].

2. Without limiting the generality, the first two equations in (1.1) can be considered homogeneous. Indeed, because of the linearity of the equations, this can be achieved by making the third equation complicated. The solution of the system (1.1) can then be sought as

$$\sigma_n(s) = \bar{\sigma}_{-n}(s) = s^n \int_s^\infty \frac{df_n(t)}{t^{2n}\sqrt{t^2 - s^2}} \tag{2.1}$$

$$\tau_{n+1}(s) = \frac{C}{s^n}\frac{d}{ds}\int_a^s \frac{f_n(t)\,dt}{\sqrt{s^2 - t^2}} - \frac{D_n}{s^{n+1}\sqrt{s^2 - a^2}}$$

$$\tau_{-n+1}(s) = \frac{C}{s^n}\frac{d}{ds}s^{2n}\int_a^s \frac{dz}{z^{2n}}\frac{d}{dz}\int_a^z \frac{\bar{f}_n(t)\,dt}{\sqrt{z^2 - t^2}} +$$

$$\frac{\bar{D}_n}{s^n}\frac{d}{ds}s^{2n}\int_0^s \frac{dz}{z^{2n+1}\sqrt{z^2 - a^2}}$$

Here f_n is the desired complex stress function, C and D_n are constants to be determined. Substituting (2.1) in the first two equations of (1.1) satisfies them identically if the following conditions are satisfied

$$(1)\quad C = \frac{\alpha}{\gamma_1\gamma_2}, \qquad (2)\quad D_n = -2n\frac{\alpha}{\gamma_1\gamma_2}a^{2n+1}\int_a^\infty \frac{f_n(t)}{t^{2n+1}}dt \tag{2.2}$$

$$(3)\quad \frac{\pi^{3/2}}{2}(G_1 + G_2)\,D_n\frac{\Gamma(n+1)}{\Gamma(n+3/2)} +$$

$$2\pi H\alpha a^{2n+2}\int_a^\infty \left[x f_n(x) - \int_a^x f_n(t)\,dt\right]x^{-2n-2}(x^2 - a^2)^{-1/2} = 0$$

Then we use the following integral which originates upon substituting (2.1) into the third of equations (1.1)

$$(1)\quad \int_a^x \frac{s^{2n+1}\,ds}{\sqrt{x^2 - s^2}}\int_s^\infty \frac{df_n(t)}{t^{2n+1}\sqrt{t^2 - s^2}} = \int_a^\infty \Big[\sqrt{x^2 - a^2}\sqrt{t^2 - a^2}\,Q_n(x,t) +$$

$$L_n(x,t)\ln\frac{\sqrt{x^2 - a^2} + \sqrt{t^2 - a^2}}{\sqrt{|t^2 - x^2|}}\Big]\frac{df_n(t)}{t^{2n}}$$

$$(2)\quad L_n(x,t) = \frac{x^{2n}}{\pi}\sum_{k=0}^n \frac{\Gamma(n+1/2-k)\,\Gamma(1/2+k)}{\Gamma(n+1-k)\,\Gamma(1+k)}\left(\frac{t}{x}\right)^{2k}$$

Here Q_n is an even polynomial in x, t, whose coefficients are determined by known recursion relationships [5].

Let us divide both sides of the third equation in (1.1) by ρ^n, let us differentiate with respect to ρ, multiply the result by $\rho^{2n}/\sqrt{\rho^2 - r^2}$ and integrate with respect to ρ between r and ∞. Use of the known properties of hypergeometric functions [5] permits separating the integral equation into singular and degenerate parts. The governing equation to determine f_n is

$$-\frac{\sqrt{r^2-a^2}}{r}\int_a^\infty \frac{tf_n(t)\,dt}{(t^2-r^2)\sqrt{t^2-a^2}}+\frac{\alpha^2}{\gamma_1\gamma_2}\int_a^\infty \frac{f_n(t)\,dt}{t^2-r^2}=\chi_n(r) \qquad (2.3)$$

Here

$$\chi_n(r)=\frac{1}{4\pi H}\int_r^\infty \frac{\rho^{2n}\,d\rho}{\sqrt{\rho^2-r^2}}\frac{d}{d\rho}\left[\frac{\Phi_n(\rho)}{\rho^n}\right]- \qquad (2.4)$$

$$\frac{2}{\pi}\int_r^\infty \frac{\rho^{2n}\,d\rho}{\sqrt{\rho^2-r^2}}\frac{d}{d\rho}\int_\rho^\infty \frac{\sqrt{x^2-a^2}\,dx}{x^{2n}\sqrt{x^2-\rho^2}}\int_a^\infty \sqrt{t^2-a^2}\,Q_n(x,t)\frac{df_n(t)}{t^{2n}}+$$

$$\frac{2}{\pi^{3/2}}\int_r^\infty \frac{\rho^{2n-1}\,d\rho}{\sqrt{\rho^2-r^2}}\int_\rho^\infty \frac{dx}{\sqrt{x^2-\rho^2}\sqrt{x^2-a^2}}\int_a^\infty \sqrt{t^2-a^2}\,R_n(\rho,x,t)\frac{df_n(t)}{t^{2n}}$$

$$R_n(\rho,x,t)=\sum_{k=0}^{n-1}\frac{\Gamma(1/2+n-k)}{\Gamma(n-k+1)}\left(\frac{t}{\rho}\right)^{2k}\times$$
$$F\left(\frac{1}{2},\ 1-n+k;\ \frac{1}{2}-n+k;\ \frac{t^2}{x^2}\right)$$

Since $n>k$, then R_n is an even polynomial, and the integrals corresponding to the degenerate part of the kernel are elementary functions for any n. The exact solution of (2.3) is [4]

$$f_n(t)=\frac{4\,\mathrm{ch}^2\,\pi\theta}{\pi^2}t\,[X_n^c(t)\,Y_c(t)+X_n^s(t)\,Y_s(t)]+A_nY_c(t) \qquad (2.5)$$

Here

$$X_n^{c,s}(t)=\int_a^\infty \frac{\chi_n(r)}{r^2-t^2}\left\{\begin{matrix} tY_c(r)\\ rY_s(r)\end{matrix}\right\}dr$$

$$Y_{c,s}(t)=\left\{\begin{matrix}\cos\\ \sin\end{matrix}\right.\left[\theta\ln\frac{t+a}{t-a}\right]$$

$$\theta=\pi^{-1}\,\mathrm{Arth}\ [\alpha\ (\gamma_1\gamma_2)^{-1/2}]$$

where A_n is an arbitrary constant which corresponds to the homogeneous solution of (2.3). The magnitudes of the constants corresponding to the degenerate part of the kernel, as well as A_n and D_n, are determined from the system of linear algebraic equations to which the two conditions (2.2) are added.

The formulated problem is solved in general form.

3. Let us consider an example. Within the circle $\rho\leqslant a$ a normal concentrated force P is applied at a distance b from the center. We assume the exterior of the circle to be rigidly clamped. Let us determine the stresses in the framing. We have a problem with boundary conditions on the boundary $z=0$

$$\sigma_z=P\delta(\rho-b)\,\delta(\varphi-0),\ \tau=0\quad (\rho\leqslant a) \qquad (3.1)$$
$$u=w=0\qquad (\rho>a)$$

For this case we determine

$$F_{n+1}(\rho)=PH\alpha b^n\rho^{-n-1},\qquad F_{-n+1}=0 \qquad (3.2)$$

$$\Phi_n(\rho) = -\frac{4H}{\pi} b^n \rho^n \int\limits_{\rho}^{\infty} \frac{dx}{x^{2n}\sqrt{x^2-\rho^2}\sqrt{x^2-b^2}}$$

Further computations are carried out separately for each harmonic. For example, for $n = 0$ we obtain

$$\sigma_0(\rho) = \int\limits_{\rho}^{\infty} \frac{df_0(t)}{\sqrt{t^2-\rho^2}} = \frac{1}{\rho}\frac{d}{d\rho}\int\limits_{\rho}^{\infty} \frac{tf_0(t)\,dt}{\sqrt{t^2-\rho^2}} \tag{3.3}$$

$$\tau_1(\rho) = \frac{\alpha}{\gamma_1\gamma_2}\frac{d}{d\rho}\int\limits_{a}^{\rho} \frac{f_0(t)\,dt}{\sqrt{\rho^2-t^2}} - \frac{D_0}{\rho\sqrt{\rho^2-a^2}} \qquad (\rho > a)$$

$$f_0(t) = \frac{2P}{\pi^3}\operatorname{ch}^2\pi\theta \int\limits_{a}^{\infty} \frac{Y_c(t)\,rY_c(r) + tY_s(t)\,Y_s(r)}{(r^2-t^2)\sqrt{r^2-b^2}}\,dr - D_0\frac{\gamma_1\gamma_2}{a\alpha}[1 - Y_c(t)] \tag{3.4}$$

$$D_0 = -\frac{P}{\pi^2}\frac{\alpha\operatorname{sh}2\pi\theta}{4\gamma_1\gamma_2\theta}\left[1 - \frac{2}{\pi}\operatorname{cth}\pi\theta\int\limits_{a}^{\infty}\frac{Y_s(r)\,dr}{\sqrt{r^2-b^2}}\right]$$

The results of calculating the first harmonic are

$$\sigma_1(\rho) = \bar{\sigma}_{-1}(\rho) = \rho\int\limits_{\rho}^{\infty}\frac{df_1(t)}{t^2\sqrt{t^2-\rho^2}} \tag{3.5}$$

$$\tau_2(\rho) = \frac{\alpha}{\gamma_1\gamma_2}\frac{1}{\rho}\frac{d}{d\rho}\int\limits_{a}^{\rho}\frac{f_1(t)\,dt}{\sqrt{\rho^2-t^2}} - \frac{D_1}{\rho^2\sqrt{\rho^2-a^2}}$$

$$\tau_0(\rho) = \frac{1}{\rho}\frac{d}{d\rho}\rho^2\left[\frac{\alpha}{\gamma_1\gamma_2}\int\limits_{a}^{\rho}\frac{dz}{z^2}\frac{d}{dz}\int\limits_{a}^{z}\frac{\bar{f}_1(t)\,dt}{\sqrt{z^2-t^2}} + \bar{D}_1\int\limits_{a}^{\rho}\frac{dz}{z^3\sqrt{z^2-a^2}}\right]$$

$$f_1(t) = \frac{4\operatorname{ch}^2\pi\theta}{\pi^2}\,t\,[X_1^c(t)\,Y_c(t) + X_1^s(t)\,Y_s(t)] + A_1Y_c(t)$$

$$\chi_1(r) = \frac{Pb}{2\pi r}\left[\frac{1}{\sqrt{r^2-b^2}} + \frac{1}{r+\sqrt{r^2-b^2}}\right] + \frac{2}{\pi}\left[1 - \frac{\sqrt{r^2-a^2}}{r}\right]B_1$$

The constants D_1, A_1, B_1 are determined from the linear system of algebraic equations

$$D_1 = -\frac{2\alpha}{\gamma_1\gamma_2}a^3\int\limits_{a}^{\infty}\frac{f_1(t)}{t^3}\,dt, \qquad B_1 = \int\limits_{a}^{\infty}\frac{\sqrt{t^2-a^2}}{t^2}\,df_1(t)$$

$$-\frac{4}{3}\frac{\pi\beta}{H\alpha}D_1 + 2\pi\int\limits_{a}^{\infty}\frac{xf_1(x) - \int\limits_{a}^{x} f_1(t)\,dt}{x^4\sqrt{x^2-a^2}}\,dx - Pb = 0$$

It should be noted that the system of stresses in the clamped part is such that its principal vector equals P exactly. This latter is easily shown by direct integration. If $b = 0$, then the problem becomes axisymmetric and the unique nonzero stress function is

$$f_0(t) = \frac{P}{2\pi}\left\{\frac{2}{\pi}\frac{\operatorname{ch}^2\pi\theta}{\operatorname{sh}\pi\theta}\frac{Y_s(t)}{t} - \frac{\operatorname{sh}2\pi\theta}{2\pi a\theta(1+\operatorname{ch}\pi\theta)}[1 - Y_c(t)]\right\}$$

For $\alpha = 0$ the solution is expressed in elementary functions

$$\sigma_n(\rho) = -\frac{P}{\pi^2}\frac{\sqrt{a^2-b^2}}{\sqrt{\rho^2-a^2}\,(\rho^2-b^2)}\left(\frac{b}{\rho}\right)^n, \quad \tau_n = 0$$

In the case of an isotropic body this latter holds for a Poisson's ratio of one-half.

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SOME PROBLEMS OF THE NONHOMOGENEOUS ELASTICITY THEORY

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Within the framework of the plane problem of the theory of elasticity, we consider the equilibrium of an elastic plane with thin different elastic inclusions, situated along a straight line. We give the formulation of the boundary value problems on the basis of the approach adopted from the theory of a thin airfoil. We present an effective method for obtaining the exact solution of a general class of problems of the above indicated type. We analyze the effect of the inclusions on the strength and we formulate criteria for the initiation of brittle fracture.

1. Formulation of the boundary value problem. In many materials which represent a practical interest, we frequently encounter thin elastic inclusions of a different material. Such are, for example, layers of graphite in cast iron, areas of oxidized metal in alloys, layers of low strength clay or sand in tectonic faults, welds, etc. The inclusions in the basic material lead to stress concentrations which affect essentially the strength properties of the material as a whole.

We consider the deformation of an unbounded, elastic, homogeneous, isotropic space with an arbitrary number of thin cylindrical inclusions of a different elastic material. Let the plane xy be some cross section of these cylinders. We assume that each of these inclusions has in the plane xy an axis of symmetry which coincides with the x -